J. Fluid Mech. (1998), *vol.* 359, *pp.* 265–280.   Printed in the United Kingdom

Reflection of gravity waves by a steep beach with a porous bed

By GILES L. BODY AND ULF T. EHRENMARK

Department CISM, London Guildhall University, 100 Minories, London EC3N 1JY, UK

(Received 14 October 1996 and in revised form 7 August)

Solutions are determined for normally incident non-breaking linear gravity waves in a perfect fluid over a porous plane bed of arbitrary slope α both with and without bed friction. For simplicity, computations are restricted to $\alpha = \pi/2m, m \in N$. Modifications to wave height transformations due to percolation and to friction are determined for a variety of slopes and coefficients. The effect on the reflection coefficient R_f is also studied and excellent qualitative agreement is found with recent work on damping and reflection by permeable structures. In particular, for a choice of parameters, the R_f response is determined in closed form.

1. Introduction

One of the earlier authors to study the effect on waves of percolation in a permeable sea bed was Putnam (1949). In that work, the modelling allowed consideration of a D'Arcy-law-type flow within the porous bed material. Both the impermeable sea floor and the extremity of the porous layer above it (sea bottom) were assumed to be horizontal and it was further assumed that a simple harmonic progressing wave potential existed at this bottom boundary, the pressure fluctuation of the wave driving dissipative currents within the porous layer. This simplified model allowed prediction of a dissipation function which when coupled with an empirical relation giving (frictionless) wave heights as a function of depth allowed a local 'power loss' to be computed. This power loss was subtracted from the frictionless power driving the waves and the new wave height estimated pointwise as the shore line is approached. Application was restricted to a shallow (1 : 300) beach and a steep (1 : 10) beach and the conclusion was that the reduction of energy due to percolation was significant in the shallow case but not in the steep case. In fact, in the former case it was found to be almost equivalent to the frictional loss.

One main aim of the present work is to investigate in the context of a non-hydrostatic linear model whether the conclusions for the steeper beach are really justified. Such a model will be required in view of the need to account for vertical accelerations, and comparison with results from, for example a shallow water theory, could only be attempted for intermediate slopes, e.g. of order 10%. Moreover, for steep beaches, we can expect a significant reflection coefficient thus giving credence to the classical bounded standing wave solution. However, in the present work, whilst restricting all applications to that fundamental solution, we shall maintain an amount of mathematical generality in the derivation by also allowing the linearly independent (logarithmically) unbounded standing wave.

The assumption of a spatially uniform pressure wave at the bottom is strictly only valid for horizontal beds and for the case of a steep impermeable beach it is well-known that the oscillation at the bed is far from uniform in the seaward coordinate. Recent work by Ehrenmark (1996) has shown, for example, the variable nature of both the propagation wavenumber and the second-order steady currents up to the breaker zone resulting from the inclination of the bed. The first order (non-breaking) flow on steep beaches is described reasonably well by the classical small-amplitude theory of e.g. Friedrichs (1948) or Stoker (1947) at least until within two or three wavelengths of the shore line, here assumed fixed for modelling purposes. At these distances Keller's (1958) theory followed by the shallow water theory (Lamb 1932, p. 254) should be adopted and this may be coupled (Miles 1990) with capillary/viscous theory very near the shore line. Ehrenmark (1996) has computed a non-breaking profile, based on asymptotic matching of the four components, which is uniformly valid to the shore line and thus provides a mechanism to absorb the large (infinite) amplitudes associated with the logarithmic singularity of the classical solution for an incoming progressive wave. This singularity is one which often prevents numerical modellers from adopting the classical solution to examine the behaviour of the flow even at considerable distances from the shore: they prefer instead to work with the conventional constant-depth Airy theory on the assumption that this is sufficiently accurate.

Notwithstanding the difficulty of a logarithmic singularity at the shore line therefore, it is proposed here to examine the effect, on the classical non-hydrostatic solution, of introducing a permeable sub-layer to displace the solid bed by a fixed angle. In contrast to the work of Putnam, we consider a two-phase interactive flow so that the effect of the percolation can be measured directly on the wave profile at the free surface rather than using a method which relies on the application of various empirical engineering formulae to determine energy losses. In earlier work, Body & Ehrenmark (1997), the authors discussed the flow when waves on the surface were coupled with a pumping mechanism at the bed which was chosen so as to simulate the effect of percolation. Although this one-phase model was somewhat more artificial than that proposed herein, in that the degree of percolation was explicit (and therefore provided an inhomogeneous Neumann condition on the bed), the solutions appeared to indicate a significant dependence between nearshore wave heights and the amplitude and wavelength of the (spatially oscillatory) pumping mechanism.

Figure 1 shows the geometry of the model. Under the assumption of uniform permeability in the sub-layer (D_1) the dynamic pressure therein will be a harmonic function. In the primary flow region (D_2) above this, the velocity potential will also be a harmonic function. The coupling between the two variables is provided by the interfacial boundary conditions which are taken to be continuity of pressure and normal velocity component. The tangential velocity component can therefore be expected to be discontinuous. Otherwise, the conditions are taken as in the classical problem. Full details of the formulation are given in § 2 and the solutions are constructed in § 3. These solutions involve a pair of difference equations which may be solved by the use of the Cauchy integral formula following the method of Peters (1952). Moreover, in the present work it is shown that, for the particular case where the permeable layer is also of angular thickness α, these curves may be determined explicitly and a simple analytical expression is given which agrees exactly with the numerically produced results. In § 4 we consider the case of small permeability. This enables a perturbation about the classical solution to be developed from which may be determined, for example, not only nearshore wave height transformations but also

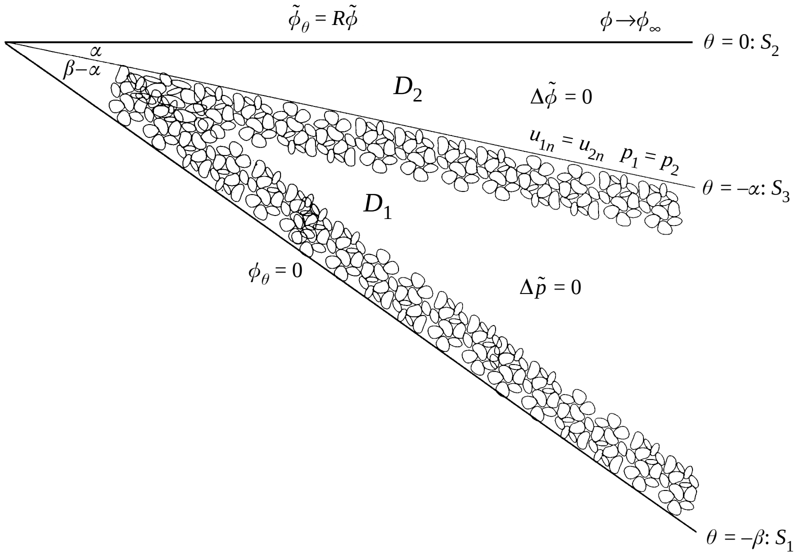


FIGURE 1. Definition diagram.

the effect of the percolation on the reflection coefficients. These are also determined for a full range of permeability values and bottom slopes and, interestingly, the shapes of the curves are, in all cases, similar to those discovered by Madsen (1983) and Mallayachari & Sundar (1994) who examined reflection from respectively a vertical permeable wave absorber and permeable seawalls. For special bottom slope angles, one of the difference equations has a closed-form solution which enables a discussion with less algebraic intricacy and makes the numerical computations easier.

In § 5 we apply the more specific model of Mallayachari & Sundar (1994) taking account of frictional effects in the porous region. This problem is also amenable to the integral transform solution and once again it is found that the reflection coefficient curves plotted against a viscosity parameter turn out to be qualitatively in agreement with those obtained analytically in Madsen (1983) using linear shallow water theory and numerically by Mallayachari & Sundar (1994) using a boundary integral method. Indeed an exact formula for the reflection coefficient is again determined for the simpler geometry case which not only vindicates the numerical technique applied in the other cases but also throws some light on the reason for the curves being of the given shape. The work is concluded with some remarks in § 6.

2. Problem formulation from D'Arcy's Law

We investigate the wave-induced flow of a perfect fluid over a porous bed. We assume the linearized Euler's equation in the fluid and use conventional Mellin transforms – defined as $F(s) = \int_0^\infty R^{s-1} f(R) \mathrm{d}R$ along with the inversion formula

$$f(R) = \frac{1}{2\pi \mathrm{i}} \int_{c_0 - \mathrm{i}\infty}^{c_0 + \mathrm{i}\infty} R^{-s} F(s) \mathrm{d}s$$

to solve the resulting mixed boundary value problem following work by Ehrenmark (1989, 1991). Specifying incoming and outgoing waves at infinity determines the flow which in general is singular at the shore line. For a unit reflection coefficient the potential is regular at the shore line in the absence of percolation.

The flow of inviscid water in irrotational motion is now modelled for small-amplitude gravity waves over a porous sloping bed. Let $\alpha > 0$ be the angle of slope of the porous bed and $\beta \in (\alpha, \pi)$ be angle of the impermeable bottom to the porous region (see figure 1 for definition of coordinate system). Velocities are assumed small so that the linearized equations are applicable.

Within the porous bed we assume that the seepage velocity $\boldsymbol{u}$ is given by D'Arcy's law

$$\boldsymbol{u} = -\kappa\nabla(p + \rho g y) \text{ for } -\beta < \theta < -\alpha.$$

Here $\kappa = K/\mu$, $\mu =$ dynamic viscosity, $K =$ permeablity, $[\mu] = ML^{-1}T^{-1}$, $[\kappa] = L^3M^{-1}T$, $[K] = L^2, [\mu/\rho] = L^2T^{-1}$. At room temperature kinematic viscosity $= \mu/\rho = 10^{-6}\,\mathrm{m^2\,s^{-1}}$, $\rho = 10^3\,\mathrm{Kg\,m^{-3}}$, $\mu = 10^{-3}\,\mathrm{Kg\,m^{-1}\,s^{-1}}$ for water. For sand in SI units $K \in (10^{-12}, 10^{-9})$, $\kappa \in (10^{-9}, 10^{-6})$ see Mei (1989, p. 685) for K for various materials – K is not a property of the fluid. Dean & Dalrymple (1991, p. 277) investigated waves over a horizontal porous bed at depth h similar to Putnam (1949) except normal velocities were taken to be continuous across the bed. In Dean & Dalrymple (1991) it was found that the free surface is given by $y = \mathrm{Re}\exp(\mathrm{i}kx - \mathrm{i}\omega t)$, $\omega^2 = gk_r \tanh k_r h$,

$$k_i = \frac{2\kappa\omega\rho k_r}{2k_r h + \sinh 2k_r h}$$

to leading order for κ small, where $k_r = \mathrm{Re}(k)$, $k_i = \mathrm{Im}(k)$. The parameter $\nu = \omega^2/g$, $[\nu] = L^{-1}$ defines a length scale for our problem.

Let $S_1 = \{(x, -x\tan\beta) : x > 0\}, S_2 = \{(x, 0) : x > 0\}, S_3 = \{(x, -x\tan\alpha) : x > 0\}$ and $D_1 = \{(x, y) : -x\tan\beta < y < -x\tan\alpha < 0\}, D_2 = \{(x, y) : -x\tan\alpha < y < 0\}$. So D_1 is porous material and D_2 is the undisturbed water. Dean & Dalrymple (1991, p. 279) state that $\boldsymbol{u}\cdot\boldsymbol{n}, p$ are continuous at S_3 (tangential component of $\boldsymbol{u}$ is discontinuous). Assume $\boldsymbol{u} = \epsilon\mathrm{Re}(\exp(\mathrm{i}\omega t)\nabla\tilde{\phi})$ in D_2, $p + \rho g y = \epsilon\mathrm{Re}(\exp(\mathrm{i}\omega t)\tilde{p})$ in D_1 where ϵ is a small ordering parameter. From a continuity equation in D_2,

$$\nabla^2\tilde{\phi} = 0 \tag{2.1}$$

and from a continuity equation in D_1

$$\nabla^2\tilde{p} = 0. \tag{2.2}$$

The linearized Euler equation is $\boldsymbol{u}_t = -\nabla p/\rho - g\boldsymbol{j}$, and we write further

$$p = -\epsilon\rho\omega\mathrm{Re}(\mathrm{i}\tilde{\phi}\exp(\mathrm{i}\omega t)) - gy\rho + w_1(t) \text{ in } D_2$$

where w_1 is an arbitrary function of t which is set to zero. At the rigid boundary S_1 we adopt the usual condition of zero normal flow, i.e. $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. Let $\tilde{\phi} = \phi_{(1)} + \mathrm{i}\phi_{(2)}, \tilde{p} = p_{(1)} + \mathrm{i}p_{(2)}$ where $\phi_{(j)}, p_{(j)}$ are real. There follows

$$\boldsymbol{u} = \begin{cases} -\kappa\epsilon(\nabla p_{(1)}\cos\omega t - \nabla p_{(2)}\sin\omega t) = -\kappa\epsilon\mathrm{Re}(\exp(\mathrm{i}\omega t)\nabla\tilde{p}) & \text{in } D_1 \\ \epsilon(\nabla\phi_{(1)}\cos\omega t - \nabla\phi_{(2)}\sin\omega t) = \epsilon\mathrm{Re}(\exp(\mathrm{i}\omega t)\nabla\tilde{\phi}) & \text{in } D_2, \end{cases}$$

$$p = \begin{cases} -\rho g y + \epsilon(p_{(1)}\cos\omega t - p_{(2)}\sin\omega t) = -\rho g y + \epsilon\mathrm{Re}(\exp(\mathrm{i}\omega t)\tilde{p}) & \text{in } D_1 \\ -\rho g y + \epsilon\omega\rho(\phi_{(1)}\sin\omega t + \phi_{(2)}\cos\omega t) = -\rho g y - \epsilon\omega\rho\mathrm{Re}(\mathrm{i}\exp(\mathrm{i}\omega t)\tilde{\phi}) & \text{in } D_2, \end{cases}$$

$$\eta = \frac{\epsilon\omega}{g}(\phi_{(1)}\sin\omega t + \phi_{(2)}\cos\omega t) = -\frac{\epsilon\omega}{g}\mathrm{Re}(\mathrm{i}\exp(\mathrm{i}\omega t)\tilde{\phi}), \text{on } S_2. \tag{2.3}$$

Here η is found by assuming constant pressure at the free surface.
The kinematic condition at S_1 leads to $p_{(1)\theta} = p_{(2)\theta} = 0$ at S_1 which implies

$$\tilde{p}_\theta = 0 \quad \text{at} \quad \theta = -\beta. \tag{2.4}$$

The continuity of $\boldsymbol{u} \cdot \boldsymbol{n}$ at S_3 leads to

$$\tilde{\phi}_\theta = -\kappa \tilde{p}_\theta \quad \text{at} \quad \theta = -\alpha. \tag{2.5}$$

The continuity of p at S_3 implies

$$\tilde{p} = -\mathrm{i}\omega\rho\tilde{\phi} \quad \text{at} \quad \theta = -\alpha. \tag{2.6}$$

The kinematic condition at S_2 gives $\eta_t = \epsilon \mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t} r^{-1} \tilde{\phi}_\theta)$ so that

$$\tilde{\phi}_\theta = r v \tilde{\phi} \quad \text{at} \quad \theta = 0. \tag{2.7}$$

We non-dimensionalize the problem given by putting $rv = R$, $\tilde{\phi} = \tilde{\phi}_{nd} g^2/\omega^3$, $\tilde{p} = \tilde{p}_{nd} \rho g^2/\omega^2$. After substituting in (2.1), (2.2), (2.4), (2.5), (2.6), (2.7) and dropping subscripts (2.1), (2.2), (2.4) are unaffected and (2.5), (2.6), (2.7) become respectively

$$\tilde{\phi}_\theta = -\hat{\epsilon} \tilde{p}_\theta \quad \text{at} \quad \theta = -\alpha, \tag{2.8}$$

$$\tilde{p} = -\mathrm{i}\tilde{\phi} \quad \text{at} \quad \theta = -\alpha, \tag{2.9}$$

$$\tilde{\phi}_\theta = R\tilde{\phi} \quad \text{at} \quad \theta = 0, \tag{2.10}$$

where $\hat{\epsilon} = \kappa\omega\rho$. Let us denote problem (2.1), (2.2), (2.4), (2.8), (2.9), (2.10) as problem (P).

3. Solution using Mellin transforms

In this section we obtain an analytic solution of (P) where $\tilde{\phi}(R, 0)$ is oscillatory at infinity. To keep the most generality we first obtain $\tilde{\phi}, \tilde{p}$ which in general are singular at the shore line. Let $P(s,\theta), \theta \in (-\beta, -\alpha)$, $\Phi(s,\theta), \theta \in (-\alpha, 0)$ be the Mellin transforms of $\tilde{p}(R,\theta), \tilde{\phi}(R,\theta)$ respectively, i.e.

$$P(s,\theta) = \int_0^\infty R^{s-1} \tilde{p}(R,\theta) \mathrm{d}R, \qquad \Phi(s,\theta) = \int_0^\infty R^{s-1} \tilde{\phi}(R,\theta) \mathrm{d}R.$$

Then from (2.4),(2.10) we obtain formally

$$\Phi = A(s) \cos s\theta + A(s+1) s^{-1} \sin s\theta, \tag{3.1}$$

$$P = A_1(s) \cos s(\theta + \beta), \tag{3.2}$$

and from (2.8),(2.9) respectively

$$sA(s) \sin s\alpha + A(s+1) \cos s\alpha = \hat{\epsilon} s A_1(s) \sin s(\beta - \alpha), \tag{3.3}$$

$$A_1(s) \cos s(\beta - \alpha) = -\mathrm{i}(A(s) \cos s\alpha - s^{-1} A(s+1) \sin s\alpha). \tag{3.4}$$

We eliminate A_1 from (3.3),(3.4) and we then substitute $A(s) = \lambda(s)\Gamma(s)d(s)c(s)$ (where $\lambda(s)$ is any period-1 function) and

$$c(s+1)/c(s) = -\tan s\alpha \tag{3.5}$$

in the resulting equation to give

$$d(s+1) = d(s)h(s), \tag{3.6}$$

where

$$h(s) = \frac{1 + \mathrm{i}\hat{\epsilon}\tan s(\beta - \alpha)\cot s\alpha}{1 - \mathrm{i}\hat{\epsilon}\tan s(\beta - \alpha)\tan s\alpha}. \tag{3.7}$$

The difference equation (3.5) has been discussed in Ehrenmark (1989, 1991) in solving the impermeable ($\kappa = 0$) bed problem. For $\alpha = \pi/(2m), m \in \boldsymbol{N}$ an explicit solution of (3.5) is

$$c(s) = 2^{m-1}\prod_{j=0}^{m-1}\cos\alpha(s + j) \tag{3.8}$$

which was given in Ehrenmark (1991); for general α, $c(s)$ is given as an integral which must be solved numerically (Ehrenmark 1991). The factor 2^{m-1} in (3.8) scales $\tilde{\phi}$ so that the wave has unit amplitude at infinity for unscaled λ (we could scale λ instead).

Writing the difference equation (3.6) as $\log(d(s+1)) - \log(d(s)) = \log(h(s))$ we can use the Cauchy formula to express a solution in the form

$$d(s) = \exp(\mathrm{i}\int_{-\infty}^{\infty}\log(h(x\mathrm{i}))/(1 - \exp(2\pi(x + \mathrm{i}s)))\mathrm{d}x),\ 0 < \mathrm{Re}(s) < 1. \tag{3.9}$$

The original difference equation (3.6) can then be used to extend the domain of definition of d. The integral expression (3.9) satisfies $\lim_{\epsilon\downarrow 0} d(1 + \mathrm{i}t - \epsilon)/d(\mathrm{i}t + \epsilon) = h(\mathrm{i}t), t \in \boldsymbol{R}$, provided $h(s)$ is Holder continuous on $\mathrm{Re}(s) = 0$. The function d defined above is analytic in the region $0 \leqslant \mathrm{Re}(s) \leqslant 1$.

From (3.1),(3.2)

$$\Phi = \Gamma\, d\,\lambda\, c\,[\cos s\theta - h(s)\sin s\theta\tan s\alpha], \tag{3.10}$$

$$P = -\mathrm{i}\Gamma\, d\,\lambda\, c\,\cos s(\theta + \beta)[\cos s\alpha + h(s)\sin^2 s\alpha\sec s\alpha]/\cos s(\beta - \alpha) \Rightarrow$$

$$P = -\frac{\mathrm{i}\Gamma\, d\,\lambda\, c\,\cos s(\theta + \beta)\sec s\alpha}{(1 - \mathrm{i}\hat{\epsilon}\tan s(\beta - \alpha)\tan s\alpha)\cos s(\beta - \alpha)}. \tag{3.11}$$

We need to choose λ so that $\tilde{\phi}(R, 0)$ has unit-amplitude oscillatory behaviour at infinity. We will make use of results like

$$\frac{1}{2\pi\mathrm{i}}\int_{c_0-\mathrm{i}\infty}^{c_0+\mathrm{i}\infty} R^{-s}\Gamma(s)\sin(s(\theta + \pi/2) + \epsilon_0)\,\mathrm{d}s = \exp(R\sin\theta)\sin(R\cos\theta + \epsilon_0),$$

where $c_0 > 0, -\pi \leqslant \theta \leqslant 0$ and ϵ_0 is arbitrary and is absolutely convergent for $-\pi < \theta < 0$ and only conditionally convergent for $\theta = 0, -\pi$. We have

$$\tilde{\phi}(R, \theta) = \phi_\infty + \frac{1}{2\pi\mathrm{i}}\int_{c_0-\mathrm{i}\infty}^{c_0+\mathrm{i}\infty}\{\Phi - \Gamma(s)\sin(s(\theta + \pi/2) + \epsilon_0)\}R^{-s}\,\mathrm{d}s,$$

where

$$\phi_\infty = \exp(R\sin\theta)\sin(R\cos\theta + \epsilon_0), \quad 0 < c_0.$$

For $\lim_{R\to\infty}\tilde{\phi}(R, 0) - \phi_\infty(R, 0) = 0$ we need

$$\Phi(s, 0) - \Gamma(s)\sin(s\pi/2 + \epsilon_0) = \Gamma(s)(d(s)\lambda(s)c(s) - \sin(s\pi/2 + \epsilon_0))$$

to be absolutely integrable on $\mathrm{Re}(s) = c_0$. We note that

$$\Gamma(c_0 + \mathrm{i}t) \sim (2\pi)^{1/2}\exp[(\mathrm{i}\pi/2)\mathrm{sgn}(t)(c_0 - 1/2)]|t|^{c_0-1/2}\exp(\mathrm{i}t\log|t|)\mathrm{e}^{-\mathrm{i}t}\mathrm{e}^{-\pi|t|/2} \text{ as } t \to \pm\infty,$$

and $c(s) \sim \sin(\frac{1}{2}\pi s + \gamma_m)$ as $\mathrm{Im}(s) \to \pm\infty$ where $\gamma_m = \frac{1}{4}\pi(1 + m)$. Let

$$d_\infty = \lim_{t\to+\infty} d(c_0 + \mathrm{i}t)$$

and note from (3.9) $\lim_{t\to-\infty} d(c_0+\mathrm{i}t)=1$, $h(c_0+\mathrm{i}\infty)=h(c_0-\mathrm{i}\infty)=1$. From (3.9)

$$d_\infty=\exp\left(\mathrm{i}\int_{-\infty}^{\infty}\log h(x\mathrm{i})\mathrm{d}x\right). \tag{3.12}$$

Hence we need $\lim_{\mathrm{Im}(s)\to+\infty}\lambda(s)=\mathrm{e}^{\mathrm{i}(\gamma_m-\epsilon_0)}d_\infty^{-1}$, $\lim_{\mathrm{Im}(s)\to-\infty}\lambda(s)=\mathrm{e}^{\mathrm{i}(\epsilon_0-\gamma_m)}$. We therefore put

$$\lambda(s)=\frac{\exp(\mathrm{i}(\epsilon_0-\gamma_m)+2\pi\mathrm{i}s)-\exp(-\mathrm{i}(\epsilon_0-\gamma_m))/d_\infty}{\exp(2\pi\mathrm{i}s)-1}.$$

The Mellin transform P with the above λ is absolutely integrable along the inversion contour. Meanwhile the Mellin transform of the normal velocity on the bed is, from (3.10),

$$\Phi_\theta(s,-\alpha)=\Gamma\, d\lambda c s(1-h(s))\sin s\alpha.$$

There is an additional requirement for (2.10) to be satisfied, namely $\Phi(s,0)$ analytic in $0<c_0\leqslant\mathrm{Re}(s)\leqslant c_0+1$. Because $\Phi(s,0)=\Gamma(s)d(s)\lambda(s)c(s)$, λ has a simple pole at 1, c has a zero at 1, Γ is analytic in the strip concerned, d is analytic in $0\leqslant\mathrm{Re}(s)\leqslant 1$, $h(s)$ analytic in $0\leqslant\mathrm{Re}(s)\leqslant c_0$ for c_0 small, a sufficient condition is to take c_0 positive and sufficiently small. A sufficient condition for c_0 is $1-\mathrm{i}\hat{\epsilon}\tan s(\beta-\alpha)\tan s\alpha\neq 0$, $\forall 0\leqslant\mathrm{Re}(s)\leqslant c_0$. The positions of the zeros and the poles of h can be found exactly in the two special cases $\beta=2\alpha$ ($h(s)=(1+\mathrm{i}\hat{\epsilon})/(1-\mathrm{i}\hat{\epsilon}\tan^2 s\alpha)$) and $\beta=3\alpha$ ($h(s)=(1+\mathrm{i}\hat{\epsilon}-\tan^2 s\alpha)/(1-(2\mathrm{i}\hat{\epsilon}+1)\tan^2 s\alpha)$). Equations (3.10),(3.11) provide a solution for any angle $\beta\in(0,\pi]$.

From §2 $\boldsymbol{u}=\mathrm{Re}(\exp(\mathrm{i}\omega t)\nabla\tilde{\phi})$(without loss of generality $\epsilon=1$) so that if the complex reflection coefficient is denoted R_f then $\tilde{\phi}(R,0)\sim C(\mathrm{e}^{\mathrm{i}R}+R_f\mathrm{e}^{-\mathrm{i}R}), R\to\infty$ (this is assuming that $\omega>0$) where C is a constant. Equivalently

$$\tilde{\phi}(R,0)\sim C(1+R_f)\cos R+\mathrm{i}C(1-R_f)\sin R. \tag{3.13}$$

Given that

$$\lambda(s)=\frac{\exp(\mathrm{i}(\epsilon_0-\gamma_m)+2\pi\mathrm{i}s)-\exp(-\mathrm{i}(\epsilon_0-\gamma_m))/d_\infty}{\exp(2\pi\mathrm{i}s)-1}\Rightarrow\tilde{\phi}(R,0)\sim\sin(R+\epsilon_0)$$

we now take one solution with $\epsilon_0=\frac{1}{2}\pi$ and another with $\epsilon_0=0$. Adding the appropriate multiples we have, for the solution which has the asymptotic behaviour (3.13),

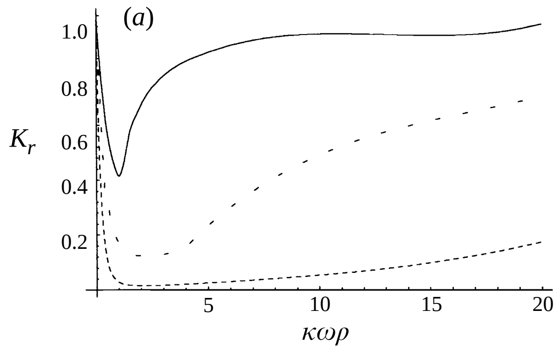
$$\begin{aligned}\lambda(s)&=(\exp(2\pi\mathrm{i}s)-1)^{-1}C[(R_f+1)(\exp(\mathrm{i}(-\gamma_m+\pi/2)+2\pi\mathrm{i}s)-\exp(\mathrm{i}\gamma_m-\mathrm{i}\pi/2)/d_\infty)\\&\quad+\mathrm{i}(1-R_f)(\exp(-\mathrm{i}\gamma_m+2\pi\mathrm{i}s)-\exp(\mathrm{i}\gamma_m)/d_\infty)]\\&\Rightarrow\lambda(s)=2\mathrm{i}C(\exp(2\pi\mathrm{i}s)-1)^{-1}(\mathrm{e}^{2\pi\mathrm{i}s-\mathrm{i}\gamma_m}+R_f\mathrm{e}^{\mathrm{i}\gamma_m}/d_\infty).\end{aligned} \tag{3.14}$$

Now by a general result on asymptotics of inverse Mellin transforms (Oberhettinger 1974, Theorem 7, p. 7) we have $\tilde{\phi}(R,\theta)$ is bounded at $R=0$ if $\Phi(s,\theta)$ has at most a simple pole at $s=0$ if $\lambda(s)$ is bounded at $s=0$ if $R_f=-\mathrm{e}^{-2\mathrm{i}\gamma_m}d_\infty$. That is from (3.12)

$$R_f=-\mathrm{e}^{-2\mathrm{i}\gamma_m}\exp\left(\mathrm{i}\int_{-\infty}^{\infty}\log h(x\mathrm{i})\,\mathrm{d}x\right). \tag{3.15}$$

The real reflection coefficient is $K_r=|R_f|=|d_\infty|$. Note that

$$|d_\infty|=\exp\left(-\int_{-\infty}^{\infty}\arg(h(x\mathrm{i}))\mathrm{d}x\right),$$



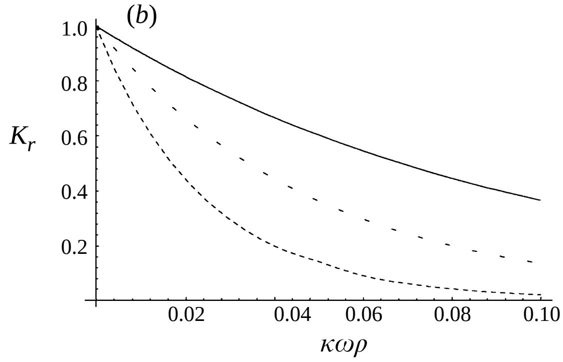


FIGURE 2. Reflection coefficients determined from (3.15) (*a*) ——, $\alpha = \pi/2$; – – –, $\alpha = \pi/4$; - - -, $\alpha = \pi/8$ and in all cases $\beta = 2\alpha$. (*b*) ——, $\alpha = \pi/16$; – – –, $\alpha = \pi/32$; - - -, $\alpha = \pi/64$ and in all cases $\beta = 2\alpha$.

where $\arg \in (-\pi, \pi)$. Note also $K_r \leqslant 1$ from the physical consideration that over-reflection is not anticipated and this can be deduced from (3.15) because $\mathrm{Im}(h(x\mathrm{i})) > 0$, $\forall x \in \boldsymbol{R}$. For the classical problem $\kappa = 0$, $d_\infty = 1$ and $R_f = -\exp(-2\mathrm{i}\gamma_m)$ for the regular standing wave, $R_f = \exp(-2\mathrm{i}\gamma_m)$ for the singular standing wave.

In figure 2 (3.15) is integrated numerically to give reflection coefficients for arbitrary permeability. There is less reflection from shallow beaches than from steep beaches because waves incident on the former tend to break. In Appendix A it is demonstrated that, for the special case $\beta = 2\alpha$, it is possible to obtain K_r explicitly. This allows us to establish also that the minimum value of K_r occurs at $\hat{\epsilon} = 2.2016$ independently of α.

4. Solution for small permeability

The parameter $\hat{\epsilon}$ ($\hat{\epsilon}$ is non-dimensional) is in the range 10^{-6}–10^{-3} for gravity waves of frequency $\omega = 1$, that is it is small so we try a regular perturbation expansion (about the classical solution). We find from a regular expansion of (3.10),(3.11)

$$h(s) = 1 + \hat{\epsilon}h_1(s) + O(\hat{\epsilon}^2), h_1(s) = \mathrm{i}\tan s(\beta - \alpha)\sec s\alpha \ \mathrm{cosec} s\alpha,$$

$$d(s) = 1 + \hat{\epsilon}d_1(s) + O(\hat{\epsilon}^2), \quad \lambda = \lambda_0 + \hat{\epsilon}\lambda_1 + O(\hat{\epsilon}^2), \quad \tilde{\phi} = \phi_0 + \hat{\epsilon}\phi_1 + O(\hat{\epsilon}^2),$$

$$d_1(s+1) - d_1(s) = h_1(s), \tag{4.1}$$

$$P(s,\theta) = P_0 + \hat{\epsilon}P_1 + O(\hat{\epsilon}^2), \quad P_0 = -\mathrm{i}\Gamma(s)\lambda_0(s)c(s)\sec s\alpha \sec s(\beta-\alpha)\cos s(\theta+\beta),$$

$$P_1 = \Gamma\lambda c\cos s(\theta+\beta)\sec s\alpha\sec s(\beta-\alpha)(-\mathrm{i}\lambda_0 d_1 + \lambda_0\tan s(\beta-\alpha)\tan s\alpha - \mathrm{i}\lambda_1),$$

$$\Phi(s,\theta) = \Phi_0 + \hat{\epsilon}\Phi_1 + O(\hat{\epsilon}^2),$$

$$\Phi_0 = \Gamma(s)\lambda_0(s)c(s)\cos s(\theta+\alpha)\sec s\alpha, \tag{4.2}$$

$$\Phi_1 = \Gamma(s)c(s)((d_1(s)\lambda_0 + \lambda_1)\cos s(\theta+\alpha)\sec s\alpha - \lambda_0 h_1(s)\sin s\theta\tan s\alpha). \tag{4.3}$$

After substituting $d_\infty = 1 + \hat{\epsilon}d_\infty^1 + O(\hat{\epsilon}^2), d_\infty^1 = \lim_{\mathrm{Im}(s)\to+\infty} d_1(s)$ into (3.14) and expanding in powers of $\hat{\epsilon}$ we find

$$\lambda_0 = 2\mathrm{i}(\mathrm{e}^{2\pi\mathrm{i}s-\mathrm{i}\gamma_m} + R_f\mathrm{e}^{\mathrm{i}\gamma_m})/(\mathrm{e}^{2\pi\mathrm{i}s}-1), \quad \lambda_1 = -2\mathrm{i}d_\infty^1 R_f\mathrm{e}^{\mathrm{i}\gamma_m}/(\mathrm{e}^{2\pi\mathrm{i}s}-1).$$

Note that $\Phi_\theta(s,-\alpha) = \hat{\epsilon}\Phi_{1\theta}(s,-\alpha) + O(\hat{\epsilon}^2)$,

$$\Phi_{1\theta}(s,-\alpha) = -h_1(s)c\Gamma\lambda_0 s\sin s\alpha = \mathrm{i}\Gamma(s)\lambda_0(s)c(s)s\tan s(\beta-\alpha)\sec s\alpha$$

which is explicit and does not involve d_1; hence we can find $\phi_{1\theta}(R,-\alpha)$. Also P_0 does not depend on d_1 so it is explicit. Because Φ_1 is absolutely convergent, $\lim_{R\to\infty}\phi_1(R,\theta) = 0$ so R_f really is the reflection coefficient which is assumed fixed. The inversion formula gives $\phi_1(R,\theta) = (1/2\pi\mathrm{i})\int_{c_0-\mathrm{i}\infty}^{c_0+\mathrm{i}\infty} R^{-s}\Phi_1(s,\theta)\mathrm{d}s$ and a similar formula for p_1 with the same c_0. The contour is chosen so that $\Phi_1(s,0)$ is regular in $c_0 \leqslant \mathrm{Re}(s) \leqslant c_0+1, \phi_1(R,\theta)$ bounded at $R=\infty$ and $\phi_1(R,\theta)$ has as weak a singularity at $R=0$ as possible.

The difference equation (4.1) has a solution

$$d_1(s) = \mathrm{i}\int_{-\infty}^{\infty} \frac{h_1(x\mathrm{i})}{1-\mathrm{e}^{2\pi(is+x)}}\,\mathrm{d}x, \quad 0 < \mathrm{Re}(s) < 1, \tag{4.4}$$

but for some special cases d_1 can be found explicitly. Suppose $\beta = n\alpha, n > 1$ is an integer. We find that $d_1(s) = (1/2m)\sum_{j=0}^{2m-1}(s+j)h_1(s+j)$ satisfies (4.1) because h_1 is $2m$-periodic. If n is odd then $d_1(s) = -\frac{1}{2}\sum_{j=0}^{m-1}h_1(s+j)$ satisfies (4.1) because $h_1(s+m) = -h_1(s)$. In particular if $n=3$ then $h_1(s) = -2\mathrm{i}\sec 2s\alpha$ and we need m even and $\lambda \equiv 1$ (regular standing wave) or m odd and $\lambda = \cot\pi s$ (singular standing wave) for $\Phi_1(s,0) = \Gamma(s)c(s)d_1(s)\lambda(s)$ to be analytic in $0 < c_0 \leqslant \mathrm{Re}(s) \leqslant c_0+1$. For all the explicit solutions $d_\infty^1 = 0$.

Two limits of interest are $\hat{\epsilon} = 0 \Rightarrow R_f = -\exp(-2\mathrm{i}\gamma_m)$ and $\hat{\epsilon} = \infty, h(\bar{s}) = \bar{h}(s)$, $K_r = 1$. The lemma given in Appendix B justifies the first-order (in $\hat{\epsilon}$) perturbation of R_f as

$$R_f = -\mathrm{e}^{-2\mathrm{i}\gamma_m}(1 - \hat{\epsilon}I + O(\hat{\epsilon}^2)). \tag{4.5}$$

Let us consider some simple special cases of the type $\alpha = \beta/n$. It is easy to work out $\beta = 2\alpha, \Rightarrow K_r = 1-(2/\alpha)\hat{\epsilon} + O(\hat{\epsilon}^2)$, $\beta = 3\alpha, \Rightarrow K_r = 1-(\pi/\alpha)\hat{\epsilon} + O(\hat{\epsilon}^2)$, $\beta = 4\alpha, \Rightarrow K_r = 1 - (\frac{2}{3} + \pi\sqrt{3}\frac{16}{27})\hat{\epsilon}/\alpha + O(\hat{\epsilon}^2)$.

Substituting

$$\tilde{\phi} = \phi_0 + \hat{\epsilon}\phi_1 + O(\hat{\epsilon}^2), \tag{4.6}$$

$\tilde{p} = p_0 + O(\hat{\epsilon})$ directly into problem (P) gives $\nabla^2\phi_i = \nabla^2 p_0 = 0$,

$$p_{0\theta}(R,-\beta) = 0, \phi_{0\theta}(R,0) = R\phi_0(R,0), \phi_{1\theta}(R,0) = R\phi_1(R,0),$$

$$\phi_{0\theta}(R,-\alpha) = 0, \quad p_{0\theta}(R,-\alpha) = -\phi_{1\theta}(R,-\alpha), \quad p_0(R,-\alpha) = -\mathrm{i}\phi_0(R,-\alpha).$$

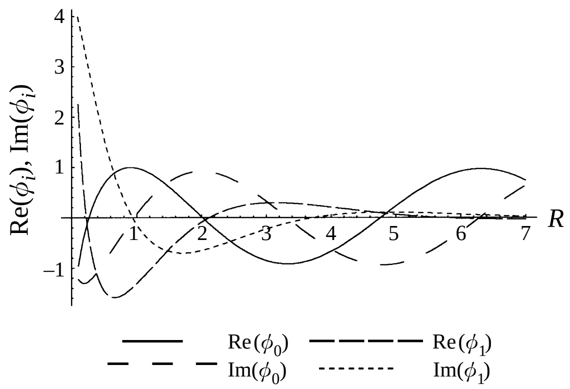


FIGURE 3. Dimensionless corrections to potentials (case of small percolation) $\alpha = \pi/8, \beta = 3\pi/8, R_f = 0$.

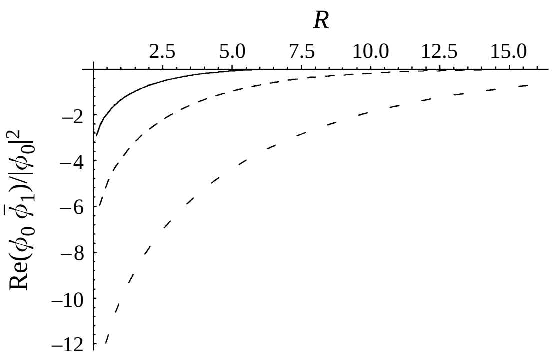


FIGURE 4. Relative reduction in wave height for small percolation, (wave height at R)/(wave height at infinity)=$1+\hat{\epsilon}Y$, ——, $\alpha = \pi/8$; - - -, $\alpha = \pi/16$; – – –, $\alpha = \pi/32$; in all cases $\beta = 3\pi/8$.

When $\mathrm{Im}(\phi_0) = 0$ (which implies a standing wave to leading order) then $\mathrm{Re}(p_0) = \mathrm{Re}(\phi_1) = 0$ and hence $\mathrm{Re}(\exp(\mathrm{i}\omega t)\tilde{p}) = -\mathrm{Im}(p_0)\sin\omega t + O(\hat{\epsilon})$,

$$\mathrm{Re}(\exp(\mathrm{i}\omega t)\tilde{\phi}) = \mathrm{Re}(\phi_0)\cos\omega t - \hat{\epsilon}\mathrm{Im}(\phi_1)\sin\omega t + O(\hat{\epsilon}^2).$$

Hence a small amount of percolation induces progressing behaviour near the shore line in a wave which has standing behaviour at infinity. Moreover, the solution which is regular at the origin will now contain an element of progressing nature at infinity. Figure 3 displays the $O(\hat{\epsilon})$ dimensionless corrections to potential induced by the percolation. Shown also are first-order potentials for comparison. The beach is chosen steep. Figure 3 is calculated by taking the inverse Mellin transforms of (4.2),(4.3) on the contour $\mathrm{Re}(s) = \frac{1}{2}$, for the case $R_f = 0$, and the function d_1 has to be tabulated using (4.4) because explicit solutions are not available due to the progressive nature of wave at infinity. The (dimensional) wave amplitude, A, from (2.3) is $A = \max\{|\eta(R,t)| : t\} = \epsilon\omega g^{-1}|\tilde{\phi}(r,0)| = v^{-1}\epsilon|\tilde{\phi}_{nd}(R,0)|$ and after substituting from (4.6) $A = \epsilon v^{-1}|\phi_0 + \hat{\epsilon}\phi_1 + O(\hat{\epsilon}^2)|$. Hence wave amplitude equals $A = \epsilon v^{-1}|\phi_0|(1 + \hat{\epsilon}\mathrm{Re}(\phi_0\bar{\phi}_1)/|\phi_0|^2) + O(\hat{\epsilon}^2)$ because $\hat{\epsilon}$ is real.

We plot $Y = \mathrm{Re}(\phi_0\bar{\phi}_1)/|\phi_0|^2$ against R in figure 4 to show the reduction in wave height due to percolation as the wave approaches the shore line, for various slope angles α.

5. Model with inertia in the bed flow

In Sollitt & Cross (1972) an alternative to D'Arcy's law to model frictional flow in porous media was derived and used to study rubble breakwaters. The fundamental governing equations for the seepage velocity are

$$u_t = -\frac{\tilde{v}}{\rho}p_x - fu, \quad v_t = -\frac{\tilde{v}}{\rho}(p + \rho g y)_y - fv,$$

where $\tilde{v}$ = porosity$\in (0, 1]$ = fraction of volume of porous medium made up of air (i.e. pores), f = bed friction, $f > 0$, $[\tilde{v}] = 1, [f] = T^{-1}$. In Mallayachari & Sundar (1994) the same equations were used to study a wave absorber with a free surface in the porous region and solutions were determined numerically using the boundary integral method. This model applies to sea walls made of rubble, not to sandy beaches. This enables evaluation of the reflection coefficient for both vertical and sloping walls. In the present work we consider instead a simplified model with no free surface in the porous region. In this way the difference equation for the Mellin transform of ϕ remains first order.

In D_2, $\boldsymbol{u} = \nabla\phi, \phi = \epsilon\mathrm{Re}(\exp(\mathrm{i}\omega t)\tilde{\phi}), p + \rho g y = -\epsilon\rho\mathrm{Re}(\exp(\mathrm{i}\omega t)\mathrm{i}\omega\tilde{\phi})$, and in $D_1, \boldsymbol{u} = \nabla\phi^p, \phi^p = \epsilon\mathrm{Re}(\exp(\mathrm{i}\omega t)\tilde{\phi}^p), p + \rho g y = \rho\tilde{v}^{-1}(-\phi^p_t - f\phi^p) = -\epsilon\rho\tilde{v}^{-1}\mathrm{Re}(\exp(\mathrm{i}\omega t)\tilde{\phi}^p(\mathrm{i}\omega + f))$ from the momentum equations. From the continuity equations,

$$\nabla^2\tilde{\phi} = 0, \quad \nabla^2\tilde{\phi}^p = 0. \tag{5.1}$$

The condition at the impermeable wall is

$$\tilde{\phi}^p_\theta = 0 \quad \text{at} \quad \theta = -\beta. \tag{5.2}$$

From continuity of $\boldsymbol{u}\cdot\boldsymbol{n}$ at S_3 we get

$$\tilde{\phi}_\theta = \tilde{\phi}^p_\theta \quad \text{at} \quad \theta = -\alpha. \tag{5.3}$$

From continuity of p at S_3 we get

$$\hat{\kappa}\tilde{\phi} = \tilde{\phi}^p \quad \text{at} \quad \theta = -\alpha, \tag{5.4}$$

where $\hat{\kappa} = \tilde{v}/(1 + f/(\mathrm{i}\omega))$ which is non-dimensional. The free-surface condition is

$$\tilde{\phi}_\theta = rv\tilde{\phi}. \tag{5.5}$$

We non-dimensionalize by $rv = R, \phi = g^2\omega^{-3}\phi_{nd}, \tilde{\phi} = g^2\omega^{-3}\tilde{\phi}_{nd}, v = \omega^2/g$ and dropping subscripts (5.1)–(5.4) are unchanged and (5.5) becomes

$$\tilde{\phi}_\theta = R\tilde{\phi} \quad \text{at} \quad \theta = 0.$$

As before, taking $\Phi, \tilde{\phi}$ as the Mellin transforms of $\tilde{\phi}, \tilde{\phi}^p$ respectively, we can write $\tilde{\phi} = A_1(s)\cos s(\theta + \beta), \Phi = A(s)\cos s\theta + s^{-1}A(s+1)\sin s\theta$, and by (5.3),(5.4) there follows

$$-sA_1(s)\sin s(\beta - \alpha) = sA(s)\sin s\alpha + \cos s\alpha\, A(s+1),$$

$$A_1(s)\cos s(\beta - \alpha) = \hat{\kappa}(A(s)\cos s\alpha - s^{-1}A(s+1)\sin s\alpha).$$

Again we write $A(s) = d(s)c(s)\Gamma(s)\lambda(s), \lambda$ of period 1, and we get $d(s+1)/d(s) = h(s)$ where

$$h(s) = \frac{1 + \hat{\kappa}\tan s(\beta - \alpha)\cot s\alpha}{1 - \hat{\kappa}\tan s(\beta - \alpha)\tan s\alpha}. \tag{5.6}$$

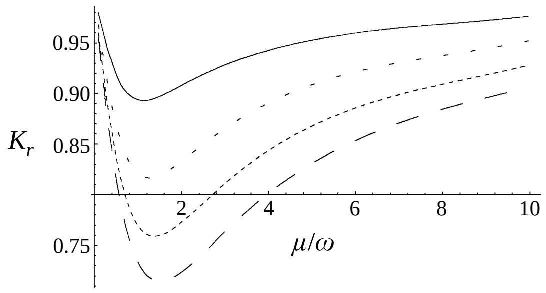


FIGURE 5. Reflection coefficients for porous model by (3.15), where h is given in (5.6), $\alpha = \pi/2, \beta = \pi$, ———, $\tilde{\nu} = 0.2$; – – –, $\tilde{\nu} = 0.4$; - - -, $\tilde{\nu} = 0.6$; ———, $\tilde{\nu} = 0.8$.

Note the similarity between (3.7) and (5.6). We get (3.10) for Φ, (3.14) for λ and

$$\begin{aligned}\tilde{\Phi} &= \cos(s(\theta+\beta))\sin s\alpha c(s)\Gamma(s)\lambda(s)(h(s)-1)d(s)/\sin(s(\beta-\alpha))\\ &= 2\hat{\kappa}\cos s(\theta+\beta)c\Gamma\lambda\sec s(\beta-\alpha)\sec s\alpha(1-\hat{\kappa}\tan s(\beta-\alpha)\tan s\alpha)^{-1}.\end{aligned}$$

As in § 3, for surface boundary condition (2.10) to be satisfied $\Phi(s,0)$ must be analytic in $c_0 \leqslant \mathrm{Re}(s) \leqslant c_0+1$ so h needs to be analytic in $0 \leqslant \mathrm{Re}(s) \leqslant c_0$. If $\beta = 2\alpha$ a sufficient condition for this is $\alpha^{-1}|\mathrm{Re}\tan^{-1}\hat{\kappa}^{-1/2}| > c_0$ which is true if c_0 is sufficiently small.

If $\tilde{\nu}$ is small then $R_f = -\exp(-2\mathrm{i}\gamma_m)(1+2\mathrm{i}\hat{\kappa}\alpha^{-1}J_{(\beta-\alpha)/(2\alpha)})+O(\hat{\kappa}^2)$ where J_λ is defined in Appendix B. We deduce that

$$K_r = 1 - 2\tilde{\nu}f\alpha^{-1}\omega J_{(\beta-\alpha)/(2\alpha)}\frac{1}{\omega^2+f^2} + O(\tilde{\nu}^2),$$

showing the dependence of K_r on f/ω seen in figure 5.

Along inversion contour $|h(s)-1| = O(\mathrm{e}^{-2\alpha|\mathrm{Im}(s)|})$ so $\tilde{\phi} = O(\mathrm{e}^{-\alpha|\mathrm{Im}(s)|}), \lim_{R\to\infty}\tilde{\phi}^p = 0$. The reflection coefficient R_f is given by (3.15) and R_f is function of $\alpha, \beta, \hat{\kappa}$ alone. There are two limits which may be interesting: $f/\omega = 0, \hat{\kappa} = \tilde{\nu}, h(\bar{s}) = \bar{h}(s), \Rightarrow K_r = 1$ and $f/\omega = \infty, \hat{\kappa} = 0 \Rightarrow R_f = -\exp(-2\mathrm{i}\gamma_m)$.

If $\beta = n\alpha$ then

$$\begin{aligned}&R_f = -\exp(-2\mathrm{i}\gamma_m)\\ &\exp(\alpha^{-1}\int_{-\infty}^{\infty}\mathrm{i}\log((1+\hat{\kappa}\tanh(n-1)w\coth w)/(1+\hat{\kappa}\tanh(n-1)w\tanh w))\mathrm{d}w).\end{aligned}$$

In figure 5 we plot K_r against f/ω showing the expected decline in K_r as f/ω increases for f/ω small as well as the decrease of K_r as $\tilde{\nu}$ increases. Similar results for K_r are found in Mallayachari & Sundar (1994) and looking at K_r as a function of ω the results in figure 5 are similar to those, using D'Arcy's law, shown in figure 2. In figure 6 we plot K_r against α showing a decrease in K_r as α decreases.

Following the approach of Appendix A, for $\beta = 2\alpha$ as assumed in figure 6, we find, after some labour,

$$K_r = \exp\left\{\frac{2}{\alpha}\mathrm{Im}\left((\tan^{-1}(\hat{\kappa}^{1/2}))^2\right)\right\}. \tag{5.7}$$

The plot of this agrees exactly with the numerically computed result.

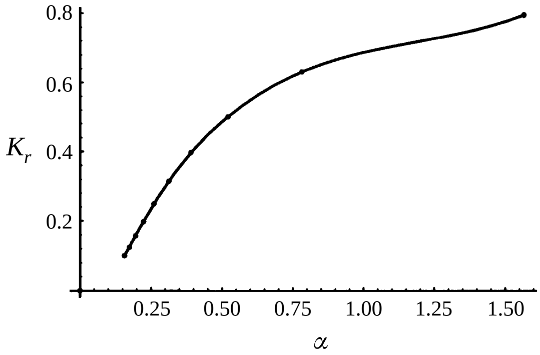


FIGURE 6. Reflection coefficients for porous model by (3.15),(5.6) $f/\omega = 2, \tilde{\nu} = 1/2, \beta = 2\alpha$.

$\tilde{\nu}$	f/ω	K_r (Mallayachari & Sundar 1994)	K_r(from (5.7))
0.2	1	0.69	0.89
0.4	1	0.50	0.82
0.6	1	0.35	0.77
0.8	1	0.22	0.73
0.5	0.25	0.33	0.91
0.5	1	0.40	0.79
0.5	3	0.6	0.84
0.5	5	0.70	0.88

TABLE 1. Comparison of K_r with Mallayachari & Sundar (1994), $\alpha = \pi/2$

6. Conclusions

The model presented herein represents a contribution to the growing number of ‘analytical’ solutions describing the response to wave attack on a beach face. Many of these models can only be used qualitatively in view of the simplifications made (the more severe of these are normally the linearization and non-breaking assumptions) but there are many recorded cases of such models performing well at least some few wavelengths from the shore line. The results from the present model appear to be in qualitative agreement with those of the parallel studies of Madsen (1983) and Mallayachari & Sundar (1994) (see table 1), and with the main perturbation to the classical model arising from a submarine alteration it should be no more affected by the two idealizations referred to above than the comparative works.

There are two responses particularly under scrutiny. On the one hand, the wave amplitude modification by bottom percolation and friction has been considered quantitatively, although here limitations of validity are somewhat more severe with one of the fundamental frictionless solution pairs being singular at the shore line. Nevertheless, for the regular wave considered, the reduction in amplitude is displayed graphically and it is found, for steep beaches, that the results of Putnam (1949) are, at least qualitatively, vindicated. Of great interest also, for obvious reasons of sea defences and induced bottom currents, is the effect of the bed configuration on the reflection coefficient of the surface waves. Extensive computations were undertaken and backed up by exact expressions available for particular geometry and the results reveal a remarkable similarity to those of the permeable sea-wall studies of Madsen (1983) and in many cases there are even greater reductions in reflection by the proposed configurations. Whether this is entirely desirable in a real situation remains open to question. Many authors (e.g. Davies & Heathershaw 1984) have studied the possibility of bottom variations increasing reflection of shorebound waves in order (possibly) to increase protection of the beach environment. On the other hand, in the case of breakwaters and absorbers, Mallayachari & Sundar (1994) argue that reflection is often undesirable as it causes standing waves leading to erosion and undermining of structures. Whichever point of view is adopted in a given situation, it is clear that there is need for further understanding of these phenomena and modifications to the present model to include more realistic bed configurations, and also bottom undulations, to enhance scattering, should be a further objective of study, where the present model could usefully provide calibration for numerical studies.

The authors are grateful to the Leverhulme Trust for funding under grant number F/405/B.

Appendix A. Exact expression for D'Arcy law reflection coefficient

In this Appendix $\tan^{-1} \in (-\pi/2, \pi/2)$. From (3.15)

$$\log K_r = -\int_{-\infty}^{\infty} \arg(h(x\mathrm{i}))\,\mathrm{d}x,$$

where

$$h(x\mathrm{i}) = \frac{1 + \mathrm{i}\hat{\epsilon}\tanh x(\beta - \alpha)\coth x\alpha}{1 + \mathrm{i}\hat{\epsilon}\tanh x(\beta - \alpha)\tanh x\alpha}.$$

Figure 2 has $\beta = 2\alpha$ so that

$$h(x\mathrm{i}) = \frac{1 + \mathrm{i}\hat{\epsilon}}{1 + \mathrm{i}\hat{\epsilon}\tanh^2 x\alpha}.$$

Thus

$$\log K_r = \int_{-\infty}^{\infty} \tan^{-1}(\hat{\epsilon}\tanh^2 x\alpha) - \tan^{-1}\hat{\epsilon}\,\mathrm{d}x.$$

We differentiate under the integral with respect to $\hat{\epsilon}$ and substitute $u = \tanh x\alpha$

$$K_r'/K_r = \frac{1}{1+\hat{\epsilon}^2}\int_{-\infty}^{\infty} \frac{(1-\tanh^2 x\alpha)(\hat{\epsilon}^2\tanh^2 x\alpha - 1)}{1+\hat{\epsilon}^2\tanh^4 x\alpha}\mathrm{d}x = \frac{\alpha^{-1}}{1+\hat{\epsilon}^2}\int_{-1}^{1}\frac{\hat{\epsilon}^2u^2-1}{1+\hat{\epsilon}^2u^4}\mathrm{d}u.$$

Hence

$$K_r'/K_r = -\frac{1}{2\alpha}(f_1f_2)', f_1 = \log\frac{1+\hat{\epsilon}+(2\hat{\epsilon})^{1/2}}{1+\hat{\epsilon}-(2\hat{\epsilon})^{1/2}},$$

$$f_2 = \begin{cases} \tan^{-1}\dfrac{(2\hat{\epsilon})^{1/2}}{1-\hat{\epsilon}} & \text{if } \hat{\epsilon} < 1 \\ \pi - \tan^{-1}\dfrac{(2\hat{\epsilon})^{1/2}}{\hat{\epsilon}-1} & \text{if } \hat{\epsilon} \geqslant 1. \end{cases}$$

Therefore

$$K_r = \exp\left(-\frac{1}{2\alpha}f_1f_2\right).$$

Appendix B. Reflection coefficient for small $\hat{\epsilon}$

LEMMA 1. *We have*

$$R_f = -\mathrm{e}^{-2\mathrm{i}\gamma_m}(1 - \hat{\epsilon}I + O(\hat{\epsilon}^2)), \quad K_r = 1 - \hat{\epsilon}I + O(\hat{\epsilon}^2),$$

where

$$I = 2\int_{-\infty}^{\infty} \tanh x(\beta - \alpha)\mathrm{cosech}2\alpha x\mathrm{d}x = \frac{1}{\alpha}J_{(\beta-\alpha)/(2\alpha)},$$

$$J_\lambda = \int_{-\infty}^{\infty} \tanh \lambda w \,\mathrm{cosech} w \,\mathrm{d}w.$$

Proof. From (3.15) it suffices to show $\mathrm{i}\int_{-\infty}^{\infty} \log h(x\mathrm{i})\mathrm{d}x = -\hat{\epsilon}I + O(\hat{\epsilon}^2)$. Note that

$$h(x\mathrm{i}) = 1 + 2\mathrm{i}\hat{\epsilon} \tanh x(\beta - \alpha)\mathrm{cosech}2\alpha x + \hat{\epsilon}^2 h_0(x, \hat{\epsilon})$$

where $h_0(x, \hat{\epsilon}) = 2(1 + \hat{\epsilon}\mathrm{i} \tanh x(\beta - \alpha) \tanh x\alpha)^{-1} \tanh^2 x(\beta - \alpha) \tanh x\alpha\mathrm{cosech}2\alpha x$. Because $\hat{\epsilon}, x \in \mathrm{R}$, we have $|h_0(x, \hat{\epsilon})| \leqslant \mathrm{sech}^2 x\alpha, \forall x \in \mathrm{R}$. For $|z| < a < 1$ $|\log(1 + z) - z| < |z|^2(1 - a)^{-1}$ so for $\hat{\epsilon}$ sufficiently small that $|h(x\mathrm{i}) - 1| < a, \forall x$ then

$$|\log h(x\mathrm{i}) - 2\mathrm{i}\hat{\epsilon} \tanh x(\beta - \alpha)\mathrm{cosech}2\alpha x| < |\hat{\epsilon}^2 h_0(x, \hat{\epsilon})| + |h(x\mathrm{i}) - 1|^2(1 - a)^{-1}, \forall x.$$

Hence

$$\left| \int_{-\infty}^{\infty} \mathrm{i} \log h(x\mathrm{i}) + 2\hat{\epsilon} \tanh x(\beta - \alpha)\mathrm{cosech}2\alpha x \,\mathrm{d}x \right| < \hat{\epsilon}^2 \int_{-\infty}^{\infty} |h_0(x, \hat{\epsilon})|\mathrm{d}x$$

$$+(1 - a)^{-1} \int_{-\infty}^{\infty} |2i\hat{\epsilon} \tanh x(\beta - \alpha)\mathrm{cosech}2\alpha x + \hat{\epsilon}^2 h_0(x, \hat{\epsilon})|^2 \mathrm{d}x$$

$$< 2\hat{\epsilon}^2\alpha^{-1} + (1 - a)^{-1} \int_{-\infty}^{\infty} 4\hat{\epsilon}^2 \tanh^2 x(\beta - \alpha)\mathrm{cosech}^2 2\alpha x + \hat{\epsilon}^4 \mathrm{sech}^4 \alpha x \,\mathrm{d}x = O(\hat{\epsilon}^2). \qquad \square$$

Note that $J_m = (\pi/m) \sum_{j=0}^{m-1} \mathrm{cosec}(\pi/m)(j + \frac{1}{2})$ if m is an integer.

REFERENCES

BODY, G. L. & EHRENMARK, U. T. 1997 The effect of pumping and ripples on the classical 2D beach problem. *Appl. Math. Lett.* (to appear).

DAVIES, A. G. & HEATHERSHAW, A. D. 1984 Surface wave propagation over sinusoidally varying topography. *J. Fluid Mech.* **144**, 419–443.

DEAN, R. G. & DALRYMPLE, R. A. 1991 *Water Wave Mechanics for Engineers and Scientists.* World Scientific.

EHRENMARK, U. T. 1989 Overconvergence of the near-field expansion for linearized waves normally incident on a sloping beach. *SIAM J. Appl. Maths* **49**, 799–815.

EHRENMARK, U. T. 1991 On viscous wave motion over a plane beach. *SIAM J. Appl. Maths* **51**, 1–19.

EHRENMARK, U. T. 1996 Eulerian mean current and Stokes Drift under non-breaking waves on a perfect fluid over a plane beach. *Fluid Dyn. Res.* **18**, 117–150.

FRIEDRICHS, K. O. 1948 Water waves on a shallow sloping beach. *Commun. Pure Appl. Maths* **1**, 109–134.

KELLER, J. B. 1958 Surface waves on water of non-uniform depth. *J. Fluid Mech.* **4**, 607–614.

LAMB, H. 1932 *Hydrodynamics*, 6th Edn. Cambridge University Press.

MADSEN, P. A. 1983 Wave reflection from a vertical permeable wave absorber. *Coastal Engng* **7**, 381–396.

MALLAYACHARI, V. & SUNDAR, V. 1994 Reflection characteristics of permeable sea walls. *Coastal Engng* **23**, 135–150.

Mei, C. C. 1989 *The Applied Dynamics of Ocean Surface Waves*. World Scientific.

Miles, J. W. 1990 Wave reflection from a gently sloping beach. *J. Fluid Mech.* **214**, 59–66.

Oberhettinger, F. 1974 *Tables of Mellin Transforms*. Springer.

Peters, A. S. 1952 Water waves over sloping beaches and the solution of a mixed boundary value problem for $\triangle\phi - k^2\phi = 0$ in a sector. *Commun. Pure Appl. Maths* **5**, 319–354.

Putnam, J. 1949 Loss of wave energy due to percolation in permeable sea bottom. *Am. Geophys. Union* **30**, 349–356.

Sollitt, C. K. & Cross, R. H. 1972 Wave transmission through permeable breakwaters. *Proc. 13th Coastal Engng Conf. Vancouver,* Vol. 3, pp. 1827–1846.

Stoker, J. J. 1947 Surface waves in water of variable depth. *Q. Appl. Maths* **5**, 1–54.